

Two applications of twisted Floer homology

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Abstract

Given an irreducible closed 3-manifold Y , we show that its twisted Heegaard Floer homology determines whether Y is a torus bundle over the circle. Another result we will prove is, if K is a genus 1 null-homologous knot in an L -space, and the 0-surgery on K is fibered, then K itself is fibered. These two results are the missing cases of earlier results due to the second author.

1 Introduction

Heegaard Floer homology was introduced by Peter Ozsváth and Zoltán Szabó in [14]. It associates to every closed oriented 3-manifold Y a package of abelian groups $\widehat{HF}(Y)$, $HF^+(Y)$, $HF^-(Y)$ and $HF^\infty(Y)$. This theory also provides an invariant, called knot Floer homology, for every null-homologous knot in a 3-manifold [17, 22].

Heegaard Floer homology turns out to be very powerful. For example, it determines the Thurston norm of a 3-manifold and the genus of a knot [19]. Ghiggini [6] and Ni [12] showed that knot Floer homology detects fibered knots. An analogue of this result for closed manifolds was proved by Ni [13], namely, Heegaard Floer homology detects whether a closed 3-manifold fibers over the circle when the genus of the fiber is greater than 1. The precise statement is as follows.

Theorem. [13] *Suppose Y is a closed irreducible 3-manifold, $F \subset Y$ is a closed connected surface of genus $g > 1$. Let $HF^+(Y, [F], g-1)$ denote the group*

$$\bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y), \langle c_1(\mathfrak{s}), [F] \rangle = 2g-2} HF^+(Y, \mathfrak{s}).$$

If $HF^+(Y, [F], g-1) \cong \mathbb{Z}$, then Y fibers over the circle with F as a fiber.

The above theorem does not hold when $g = 1$. In this case the group $HF^+(Y, [F], 0)$ is always an infinitely generated group, so its Euler characteristic

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is not well-defined in the usual sense. Ozsváth and Szabó suggested us that one may use Heegaard Floer homology with twisted coefficients in some Novikov ring. Some calculations for torus bundles has been done in an earlier paper [1] along this line.

As in [15], there is a twisted Heegaard Floer homology $\underline{HF}^+(Y; \Lambda_\omega)$, where Λ_ω is the universal Novikov ring equipped with a $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module structure which will be defined in Subsection 2.1. In [1] this twisted Heegaard Floer homology is calculated for torus bundles.

Theorem 1.1. [1] *Suppose $\pi: Y \rightarrow S^1$ is a fiber bundle with torus fiber F , and $\omega \in H^2(Y; \mathbb{Z})$ is a cohomology class such that $\omega([F]) \neq 0$. Then we have an isomorphism of Λ -modules*

$$\underline{HF}^+(Y; \Lambda_\omega) \cong \Lambda.$$

The above theorem completes in some sense a result of Ozsváth and Szabó [16] which states that a surface bundle over circle has monic Heegaard Floer homology.

In the current paper, we prove the converse to the above theorem. Our main result is:

Theorem 1.2. *Suppose Y is a closed irreducible 3-manifold, $F \subset Y$ is an embedded torus. If there is a cohomology class $\omega \in H^2(Y; \mathbb{Z})$ such that $\omega([F]) \neq 0$ and $\underline{HF}^+(Y; \Lambda_\omega) \cong \Lambda$, then Y fibers over the circle with F as a fiber.*

Remark 1.3. The proof of Theorem 1.2 is based on Ghiggini's argument in [6]. The only new ingredient we introduce here is twisted coefficients. In the setting of Monopole Floer homology, a corresponding version of this theorem was proved in [10, Theorem 42.7.1], following Ghiggini's argument.

Besides Theorem 1.2, we give an application of Theorem 1.1.

Theorem 1.4. *Suppose Y is an L -space, $K \subset Y$ is a genus 1 null-homologous knot. If the 0-surgery on K fibers over S^1 , then K itself is a fibered knot.*

The case where K has genus greater than 1 has been proved in [12, Corollary 1.4].

This paper is organized as follows. In Section 2, we collect some preliminary results on Heegaard Floer homology with an emphasis on twisted coefficients. In Section 3, we prove a key proposition which relates the Euler characteristic of ω -twisted Heegaard Floer homology with Turaev torsion. With the help of this proposition, we can prove a homological version of the main theorem. In Section 4, we give an proof of Theorem 1.2 following Ghiggini's argument. In Section 5, we prove Theorem 1.4 by using Theorem 1.1 and the exact sequence for ω -twisted Floer homology from [1].

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2 Preliminaries on Heegaard Floer Homology

We review some of the constructions in Heegaard Floer homology which will be used throughout this paper. The details can be found in [15, 17, 19, 18, 1].

2.1 Heegaard Floer homology with twisted coefficients

Let Y be a closed oriented 3-manifold and \mathfrak{t} be a Spin^c structure over Y . Ozsváth and Szabó [15] defined a universally twisted chain complex $\underline{CF}^\circ(Y, \mathfrak{t})$ with coefficients in the group ring $\mathbb{Z}[H^1(Y; \mathbb{Z})]$. Its homology $\underline{HF}^\circ(Y, \mathfrak{t})$ is an invariant of the pair (Y, \mathfrak{t}) . Furthermore, given any module M over $\mathbb{Z}[H^1(Y; \mathbb{Z})]$, they defined Floer homology with coefficients twisted by M :

$$\underline{HF}^\circ(Y, \mathfrak{t}; M) = H_*(\underline{CF}^\circ(Y, \mathfrak{t}) \otimes_{\mathbb{Z}[H^1(Y; \mathbb{Z})]} M)$$

This construction recovers the ordinary Heegaard Floer homology if we take M to be the trivial $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module \mathbb{Z} .

A special twisted Floer homology is used to investigate torus bundles over the circle in [1]. Consider the universal Novikov ring [11, Section 11.1]

$$\Lambda = \left\{ \sum_{r \in \mathbb{R}} a_r t^r \mid a_r \in \mathbb{R}, \#\{a_r \mid a_r \neq 0, r \leq c\} < \infty \text{ for any } c \in \mathbb{R} \right\}.$$

Λ itself is a field. Given a cohomology class $\omega \in H^2(Y; \mathbb{R})$, there is a group homomorphism

$$\begin{aligned} H^1(Y; \mathbb{Z}) &\rightarrow \mathbb{R} \\ h &\mapsto \langle h \cup \omega, [Y] \rangle, \end{aligned}$$

which then induces a ring homomorphism

$$\begin{aligned} \mathbb{Z}[H^1(Y; \mathbb{Z})] &\rightarrow \Lambda \\ \sum a_h \cdot h &\mapsto \sum a_h \cdot t^{\langle h \cup \omega, [Y] \rangle}. \end{aligned}$$

In this way we can equip Λ with an induced $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module structure. We denote this module by Λ_ω .

This $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module Λ_ω gives rise to a twisted Heegaard Floer homology $\underline{HF}^+(Y; \Lambda_\omega)$ which is called the ω -twisted Heegaard Floer homology. More

precisely, it is defined as follows. (For more details, see [19] and [1].) Choose an admissible pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for Y . Every Whitney disk $\phi : \mathbb{D}^2 \rightarrow \text{Sym}^g(\Sigma)$ gives rise to a two-chain in Y . Let η be a closed 2-cochain that represents ω . The evaluation of η on ϕ is denoted $\int_{\phi} \eta$. Take the $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module freely generated by all the pairs $[\mathbf{x}, i]$ (where $\mathbf{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$ and the integer $i \geq 0$), its tensor product with the module Λ_{ω} is the ω -twisted chain complex $\underline{CF}^+(Y; \Lambda_{\omega})$. We define the differential on the complex by the formula:

$$\partial^+[\mathbf{x}, i] = \sum_{\mathbf{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi)=1\}} \# \widehat{\mathcal{M}}(\phi)[\mathbf{y}, i - n_z(\phi)] \cdot t^{\int_{\phi} \eta}.$$

The homology of this chain complex only depends on the cohomology class ω . We call this homology the ω -twisted Heegaard Floer homology $\underline{HF}^+(Y; \Lambda_{\omega})$. This homology also has a Λ -module structure. Since Λ is a field, a Λ -module is actually a vector space over Λ .

Theorem 1.1 shows that this ω -twisted Floer homology is very simple for torus bundles over S^1 when ω evaluates non-trivially on the fiber. The key ingredients in the proof are an exact sequence for ω -twisted Heegaard Floer homology and the adjunction inequality.

Theorem 2.1. [1] *Let $K \subset Y$ be a framed knot in a 3-manifold Y and $\gamma \subset Y - K$ be a simple closed curve in the knot complement. For every rational number r , γ is a curve in the surgery manifold $Y_r(K)$. Let $\omega_r = \text{PD}(\gamma) \in H^2(Y_r(K); \mathbb{R})$ and $\omega = \text{PD}(\gamma) \in H^2(Y; \mathbb{R})$. Then we have the following exact sequence:*

$$\begin{array}{ccc} \underline{HF}^+(Y; \Lambda_{\omega}) & \xrightarrow{\quad} & \underline{HF}^+(Y_0(K); \Lambda_{\omega_0}) \\ & \nwarrow \quad \nearrow & \\ & \underline{HF}^+(Y_1(K); \Lambda_{\omega_1}) & \end{array} \quad (1)$$

The maps in the above sequence are induced from cobordism.

The proof of the above theorem in [1] can be applied to general integer surgeries. As in [15, Theorem 9.19], suppose \mathfrak{s} is a Spin^c structure on Y , \mathfrak{s}_k is any one of the p Spin^c structures on $Y_p(K)$ which are Spin^c -cobordant to \mathfrak{s} , $Q : \text{Spin}^c(Y_0) \rightarrow \text{Spin}^c(Y_p)$ is a surjective map. Let

$$\underline{HF}^+(Y_0(K), [\mathfrak{s}_k]; \Lambda_{\omega_0}) = \bigoplus_{\mathfrak{t} \in \text{Spin}^c(Y_0(K)), Q(\mathfrak{t})=\mathfrak{s}_k} \underline{HF}^+(Y_0(K), \mathfrak{t}; \Lambda_{\omega_0}).$$

The following exact sequence will be used in the proof of Theorem 1.4.

Theorem 2.2. *Notation is as in Theorem 2.1. For each positive integer p , the following sequence*

$$\begin{array}{ccc} \underline{HF}^+(Y, \mathfrak{s}; \Lambda_{\omega}) & \xrightarrow{\quad} & \underline{HF}^+(Y_0(K), [\mathfrak{s}_k]; \Lambda_{\omega_0}) \\ & \nwarrow \quad \nearrow & \\ & \underline{HF}^+(Y_p(K), \mathfrak{s}_k; \Lambda_{\omega_p}) & \end{array}$$

F_3^+

is exact. Here the map F_3^+ is induced by a two-handle cobordism connecting $Y_p(K)$ to Y .

Combine Theorem 2.1 and [1, Proposition 2.2], we can easily get the following corollary, which is used implicitly in [1] to prove Theorem 1.1.

Corollary 2.3. *Suppose F is an embedded torus in a closed manifold Y and $\omega \in H^2(Y; \mathbb{Z})$ is a cohomology class such that $\omega([F]) \neq 0$. Let Y' be the manifold obtained by cutting open Y along F and regluing by a self-homeomorphism of F . There is a cobordism $W: Y \rightarrow Y'$ obtained by adding 2-handles along knots in F . Take a closed curve $\gamma \subset Y$ missing the attaching regions of the 2-handles and $\text{PD}(\gamma) = \omega \in H^2(Y; \mathbb{Z})$. As in Theorem 2.1, γ also determines a cohomology class ω' in Y' by setting $\omega' = \text{PD}_{Y'}(\gamma)$. Then the map induced by cobordism is an isomorphism*

$$\underline{HF}^+(Y'; \Lambda_{\omega'}) \cong \underline{HF}^+(Y; \Lambda_{\omega}).$$

2.2 Knot Floer homology

Suppose $K \subset Y$ is a null-homologous knot in a rational homology 3-sphere, F is a fixed Seifert surface. There is a compatible doubly pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ for the knot K as in [17]. This gives rise to a map from intersection points between the two tori $\mathbb{T}_\alpha, \mathbb{T}_\beta$ to relative Spin^c structures on $Y - K$

$$s_{w,z}: \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow \underline{\text{Spin}}^c(Y, K) = \text{Spin}^c(Y_0(K)).$$

For each Spin^c structure \mathfrak{s} on Y , the knot chain complex

$$C(\mathfrak{s}) = CFK^\infty(Y, K; \mathfrak{s})$$

is a free abelian group generated by $[\mathbf{x}, i, j] \in (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z} \times \mathbb{Z}$, such that $s_{w,z}(\mathbf{x})$ extends \mathfrak{s} and

$$\frac{\langle c_1(s_{w,z}(\mathbf{x})), [\widehat{F}] \rangle}{2} + (i - j) = 0,$$

and endowed with the differential

$$\partial[\mathbf{x}, i, j] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi) = 1\}} \# \widehat{\mathcal{M}}(\phi) [\mathbf{y}, i - n_w(\phi), j - n_z(\phi)].$$

This complex is given a filtration function $\mathcal{F}[\mathbf{x}, i, j] = (i, j)$. The forgetful map $[\mathbf{x}, i, j] \mapsto [\mathbf{x}, i]$ induces an isomorphism between $C(\mathfrak{s})$ and $CF^\infty(Y, \mathfrak{s})$, sending $B_{\mathfrak{s}}^+ := C(\mathfrak{s})\{i \geq 0\}$ isomorphically to $CF^+(Y, \mathfrak{s})$. For each integer d , we define

$$\widehat{HFK}(Y, K, \mathfrak{s}; d) = H_*(C(\mathfrak{s})\{i = 0, j \leq d\} / C(\mathfrak{s})\{i = 0, j \leq d - 1\}, \partial).$$

Define $A_{\mathfrak{s},k}^+ = C(\mathfrak{s})\{\max(i, j - k) \geq 0\}$. There are two canonical chain maps $v_{\mathfrak{s},k}^+: A_{\mathfrak{s},k}^+ \rightarrow B_{\mathfrak{s}}^+$ and $h_{\mathfrak{s},k}^+: A_{\mathfrak{s},k}^+ \rightarrow B_{\mathfrak{s}}^+$. The map $v_{\mathfrak{s},k}^+$ is the projection onto $C(\mathfrak{s})\{i \geq 0\}$, while $h_{\mathfrak{s},k}^+$ is the projection onto $C(\mathfrak{s})\{j \geq k\}$, followed by the identification with $C(\mathfrak{s})\{j \geq 0\}$ induced by the multiplication by U^k , followed by the chain homotopy equivalence from $C(\mathfrak{s})\{j \geq 0\}$ to $C(\mathfrak{s})\{i \geq 0\}$.

Theorem 2.4 (Ozsváth–Szabó, [20, 21]). *Let $K \subset Y$ be a null-homologous knot in a rational homology sphere. There is an integer N with the property that for all $m \geq N$ and all $t \in \mathbb{Z}/m\mathbb{Z}$, $CF^+(Y_m(K), \mathfrak{s}_t)$ is represented by the chain complex $A_{\mathfrak{s},k}^+ = C(\mathfrak{s})\{\max(i, j-k) \geq 0\}$ where $k \equiv t \pmod{m}$ and $|k| \leq \frac{m}{2}$, in the sense that there are isomorphisms*

$$\Psi_{m,k}^+ : CF^+(Y_m(K), \mathfrak{s}_t) \rightarrow A_{\mathfrak{s},k}^+.$$

Moreover, if \mathfrak{x}_k and \mathfrak{y}_k denote the Spin^c structures over the cobordism

$$W'_m(K) : Y_m(K) \rightarrow Y$$

with

$$\langle c_1(\mathfrak{x}_k), [\widehat{F}] \rangle + m = 2k \text{ and } \langle c_1(\mathfrak{y}_k), [\widehat{F}] \rangle - m = 2k,$$

respectively. Then v_k^+ and h_k^+ correspond to the maps induced by the cobordism $W'_m(K)$ endowed with the Spin^c structures \mathfrak{x}_k and \mathfrak{y}_k , respectively.

2.3 Ozsváth–Szabó contact invariant in twisted Floer homology

Ozsváth and Szabó [18] defined an invariant $c(\xi) \in \widehat{HF}(-Y)$ for every contact structure ξ on a closed 3-manifold Y . It is defined up to sign and lies in the summand $\widehat{HF}(-Y, \mathfrak{t}_\xi)$ corresponding to the canonical Spin^c structure \mathfrak{t}_ξ associated to ξ .

In [19], Ozsváth and Szabó point out that the contact invariant can be defined for twisted Heegaard Floer homology. In fact, for any module M over $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ one can get an element

$$c(\xi; M) \in \widehat{HF}(-Y, \mathfrak{t}_\xi; M) / \mathbb{Z}[H^1(Y, \mathbb{Z})]^\times.$$

This is an element $c(\xi; M) \in \widehat{HF}(-Y, \mathfrak{t}_\xi; M)$ which is well-defined up to an overall multiplication by a unit in the group ring $\mathbb{Z}[H^1(Y; \mathbb{Z})]$. Let $c^+(\xi; M)$ denote the image of $c(\xi; M)$ under the natural map $\widehat{HF}(-Y, \mathfrak{t}_\xi; M) \rightarrow HF^+(-Y, \mathfrak{t}_\xi; M)$.

In particular, for ω -twisted Heegaard Floer homology $HF^+(Y; \Lambda_\omega)$, one gets a contact invariant $c^+(\xi; \Lambda_\omega) \in HF^+(-Y, \mathfrak{t}_\xi; \Lambda_\omega)$. It is well-defined up to multiplication by a term $\pm t^n$ for some $n \in \mathbb{R}$. There is also a non-vanishing theorem for weakly fillable contact structures in this ω -twisted version.

Theorem 2.5. [19, Theorem 4.2] *Let (W, Ω) be a weak filling of a contact manifold (Y, ξ) , where $\Omega \in H^2(W; \mathbb{R})$ is the symplectic 2-form. Then the contact invariant $c^+(\xi; \Lambda_{\Omega|_Y})$ is non-trivial.*

Remark 2.6. In [19] Ozsváth and Szabó proved that if (W, Ω) is a symplectic filling of a contact structure (Y, ξ) , then the contact invariant $c^+(\xi; [\Omega|_Y])$ is non-trivial. That proof works for our ω -twisted version as well given the following description of the ω -twisted contact invariant.

Take an open book decomposition (Y, K) compatible with the contact structure ξ . After positive stabilization, we can assume that the open book has connected binding and genus $g > 1$. Suppose $W: Y \rightarrow Y_0(K)$ is the corresponding Giroux 2-handle cobordism [7], and $\beta \in H^2(W; \mathbb{R})$ is any cohomology class on W which extends ω . Then we have

$$\underline{HF}^+(-Y_0(K), g-1; \Lambda_{\beta|_{-Y_0(K)}}) \cong \Lambda.$$

Moreover, the contact invariant $c^+(\xi; \Lambda_\omega)$ is equal to the image of $1 \in \Lambda$ under the map induced by cobordism

$$\underline{F}_W^+[\beta]: \underline{HF}^+(-Y_0(K), g-1; \Lambda_{\beta|_{-Y_0(K)}}) \rightarrow \underline{HF}^+(-Y, \mathfrak{t}_\xi; \Lambda_\omega).$$

In the untwisted case, such description for the contact invariant is proved in [18, Proposition 3.1]. In that proof, one constructed a Heegaard diagram for $Y_0(K)$ which is admissible with respect to all the Spin^c structures \mathfrak{t} such that

$$\langle c_1(\mathfrak{t}), [\widehat{F}] \rangle = 2g - 2.$$

In this diagram there are only 2 intersection points representing Spin^c structures satisfying the above restriction. Notice we can also use this Heegaard diagram to compute the ω -twisted Heegaard Floer homology. So this argument can be used to prove the above statements.

The following corollary is a little modification of [6, Lemma 2.3].

Corollary 2.7. *Suppose Y is a closed, connected, oriented 3-manifold with $b_1(Y) = 1$, and F is a closed surface in Y . Suppose ξ is a weakly fillable contact structure on Y , such that $\xi|_F$ is C^0 -close to the oriented tangent plane field of F , and $\omega \in H^2(Y; \mathbb{R})$ is a cohomology class such that $\omega([F]) > 0$. Then the ω -twisted contact invariant $c^+(\xi; \Lambda_\omega)$ is non-zero.*

Proof. Let (W, Ω) be a weak filling of the contact structure (Y, ξ) , $[\Omega] \in H^2(W)$ be the cohomology class represented by the closed 2-form Ω . By the definition of weak fillability, Ω is positive on each plane in ξ , so $\langle [\Omega]|_Y, [F] \rangle > 0$. Since $b_1(Y) = 1$ and $\omega([F]) > 0$, we may assume $[\Omega]|_Y = k\omega \in H^2(Y; \mathbb{R})$ for some positive real number k . By Theorem 2.5, $c^+(\xi; \Lambda_\Omega) \neq 0$. Notice that the map $t \mapsto t^k$ induces an isomorphism between chain complexes

$$\underline{CF}^+(-Y; \Lambda_\omega) \rightarrow \underline{CF}^+(-Y; \Lambda_\Omega),$$

it thus induces an isomorphism between homology groups

$$\underline{HF}^+(-Y; \Lambda_\omega) \rightarrow \underline{HF}^+(-Y; \Lambda_\Omega).$$

Under this isomorphism $c^+(\xi; \Lambda_\omega)$ is taken to $c^+(\xi; \Lambda_\Omega)$. This forces the contact invariant $c^+(\xi; \Lambda_\omega)$ to be non-zero. \square

Given a contact manifold (Y, ξ) and a Legendrian knot $K \subset Y$, we can do contact $(+1)$ -surgery to produce a new contact manifold $(Y_1(K), \xi')$. In [18, 9], it is showed that the untwisted contact invariant behaves well with respect to contact $(+1)$ -surgery. Similarly, the ω -twisted contact invariants are related as follows.

Proposition 2.8. [9, Theorem 2.3] *Suppose (Y', ξ') is obtained from (Y, ξ) by contact $(+1)$ -surgery along a Legendrian knot $K \subset Y$. Suppose $\gamma \subset Y - K$ is a closed curve, denote $\omega = PD(\gamma) \in H^2(Y; \mathbb{R})$. View γ as a curve in the surgery manifold Y' and denote its Poincaré dual by $\omega' \in H^2(Y'; \mathbb{R})$. Let $-W$ be the cobordism obtained from $Y \times I$ by adding a 2-handle along K with $+1$ framing and with orientation reversed. Then*

$$\underline{E}_{-W; PD(\gamma \times I)}^+(c^+(Y, \xi; \Lambda_\omega)) = c^+(Y', \xi'; \Lambda_{\omega'}).$$

3 Euler characteristic of ω -twisted Floer homology

The goal of this section is to prove a homological version of Theorem 1.2. In order to do so, we first study the Euler characteristic of twisted Heegaard Floer homology.

In [15] Ozsváth and Szabó prove that the Euler characteristic of $HF^+(Y, \mathfrak{s})$ is equal to the Turaev torsion when \mathfrak{s} is a non-torsion Spin^c structure. More precisely, they prove the following:

Theorem. [15, Theorem 5.11, Theorem 5.2] *If \mathfrak{s} is a non-torsion Spin^c structure over an oriented 3-manifold Y with $b_1(Y) \geq 1$, then $HF^+(Y, \mathfrak{s})$ is finitely generated and*

$$\chi(HF^+(Y, \mathfrak{s})) = \pm T(Y, \mathfrak{s}).$$

When $b_1(Y) = 1$, the Turaev torsion in the above equation is defined with respect to the chamber containing $c_1(\mathfrak{s})$.

Remark 3.1. The proof of the above theorem can be modified to show that when \mathfrak{s} is a non-torsion Spin^c structure, the ω -twisted Heegaard Floer homology $\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)$ is a finitely generated vector space over Λ , and

$$\chi(\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)) = \pm T(Y, \mathfrak{s}).$$

For a torsion Spin^c structure \mathfrak{s} , however, $HF^+(Y, \mathfrak{s})$ is infinitely generated. Ozsváth and Szabó [15] introduce truncated Euler characteristic for torsion Spin^c structures and prove when $b_1(Y) = 1$ or 2 , it is related to the Turaev torsion.

When we use ω -twisted Heegaard Floer homology, the situation is much simpler. In fact we have the following:

Proposition 3.2. *Suppose \mathfrak{s} is a torsion Spin^c structure over an oriented 3-manifold Y with $b_1(Y) \geq 1$. Then for any non-zero cohomology class $\omega \in H^2(Y; \mathbb{R})$, the twisted Floer homology $\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)$ is a finitely generated vector space over Λ and*

$$\chi(\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)) = \pm T(Y, \mathfrak{s}).$$

Here the Euler characteristic is taken by viewing $\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)$ as a vector space over Λ .

Notice when $b_1(Y) = 1$ and \mathfrak{s} is a torsion Spin^c structure, $T(Y, \mathfrak{s})$ does not depend on the choice of a chamber, see [23]. The following lemma is an analogue of [8, Corollary 8.5].

Lemma 3.3. *Let \mathfrak{s} be a torsion Spin^c structure on a 3-manifold Y , $\omega \in H^2(Y; \mathbb{R})$ be a non-zero cohomology class. Then*

$$\underline{HF}^\infty(Y, \mathfrak{s}; \Lambda_\omega) \cong 0.$$

Proof. It is shown in [15, Theorem 10.12] that for each torsion Spin^c structure \mathfrak{s} , the universally twisted Heegaard Floer homology

$$\underline{HF}^\infty(Y, \mathfrak{s}) \cong \mathbb{Z}[U, U^{-1}],$$

and it is a trivial $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ -module. Then there is a universal coefficients spectral sequence converging to the ω -twisted Floer homology $\underline{HF}^\infty(Y, \mathfrak{s}; \Lambda_\omega)$, and its E_2 term is given by

$$\text{Tor}_{\mathbb{Z}[H^1(Y; \mathbb{Z})]}^i(\underline{HF}_j^\infty(Y, \mathfrak{s}); \Lambda_\omega).$$

Notice $\underline{HF}_j^\infty(Y, \mathfrak{s}) = 0$ or \mathbb{Z} . There is a free resolution of \mathbb{Z} as a module over $\mathbb{A} = \mathbb{Z}[H^1(Y; \mathbb{Z})]$, given by

$$\otimes_{i=1}^{b_1(Y)} (\mathbb{A} \xrightarrow{e^{h_i}-1} \mathbb{A}),$$

where h_i is a basis for $H^1(Y; \mathbb{Z})$, and e^{h_i} is the corresponding element in the group ring \mathbb{A} (see [16, Lemma 2.3]). So the E_2 term of the above spectral sequence is calculated by the homology of

$$\otimes_{i=1}^{b_1(Y)} (\Lambda_\omega \xrightarrow{t^{d_i}-1} \Lambda_\omega),$$

where $d_i = \langle h_i \cup \omega, [Y] \rangle$. By assumption $\omega \neq 0 \in H^2(Y; \mathbb{R})$, at least one of d_i is non-zero. For this i , the map

$$\Lambda_\omega \xrightarrow{t^{d_i}-1} \Lambda_\omega$$

is an isomorphism, hence the corresponding complex has zero homology. From this we see that all the E_2 terms are 0, so are the E_∞ terms. This proves the lemma. \square

Recall that for a torsion Spin^c structure \mathfrak{s} , there is an absolute \mathbb{Q} -grading on $\underline{HF}^+(Y, \mathfrak{s})$ which lifts the relative \mathbb{Z} -grading defined in [15], see [20]. This is also the case for our ω -twisted Floer homology $\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)$. Suppose the absolute grading is supported in $\mathbb{Z} + d$ for some constant $d \in \mathbb{Q}$. The Euler characteristic of $\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)$ is defined to be

$$\chi(\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)) = \sum_{n \in \mathbb{Z}} (-1)^n \text{rank } \underline{HF}_{d+n}^+(Y, \mathfrak{s}; \Lambda_\omega).$$

Notice our d is unique up to adding an integer, so the Euler characteristic is defined up to sign.

This absolute \mathbb{Q} -grading and Lemma 3.3 lead to the following corollary.

Corollary 3.4. *Let \mathfrak{s} be a torsion Spin^c structure on a 3-manifold Y , $\omega \in H^2(Y; \mathbb{R})$ be a non-zero cohomology class. Then for all sufficiently large $N \in \mathbb{Z}$,*

$$\underline{HF}_{d+N}^+(Y, \mathfrak{s}; \Lambda_\omega) = 0.$$

Proof. The elements in $\underline{HF}^-(Y, \mathfrak{s}; \Lambda_\omega)$ have absolute \mathbb{Q} -grading bounded from above. So by the exact sequence relating \underline{HF}^- , \underline{HF}^∞ and \underline{HF}^+ we see that for all sufficiently large $N \in \mathbb{Z}$,

$$\underline{HF}_{d+N}^+(Y, \mathfrak{s}; \Lambda_\omega) \cong \underline{HF}_{d+N}^\infty(Y, \mathfrak{s}; \Lambda_\omega),$$

which is zero by Lemma 3.3. \square

Proof of Proposition 3.2. (Compare the proof of [15, Theorem 5.2].)

From Corollary 3.4, $\underline{HF}^+(Y, \mathfrak{s}; \Lambda_\omega)$ is a finitely generated vector space over Λ . So we can talk about its Euler characteristic. As in [15, Section 5.3] we can construct a Heegaard diagram $(\Sigma, \alpha, \beta, z)$ for Y such that there is a periodic domain P_1 containing α_1 with multiplicity one in its boundary. Extend $\{P_1\}$ to a basis $\{P_1, P_2, \dots, P_b\}$ for periodic domains such that P_2, \dots, P_b does not contain α_1 in their boundaries. This can be achieved by adding proper multiples of P_1 to each P_i . Choose a set of dual simple closed curves $\{a_i^*\}$ for $\{\alpha_i\}$, namely, a_i^* meets α_i transversely in a single point and misses all other α_j . Wind α_1 along a_1^* n times and put the base point z in this winding region, to the right of a_1^* and in the $\frac{n}{2}$ th subregion of the winding region. (See [15, Figure 6].) Wind $\alpha_2, \dots, \alpha_g$ along a_2^*, \dots, a_g^* sufficiently many times, such that any nontrivial linear combination of P_2, \dots, P_n has both large positive and negative local multiplicities as in [14, Lemma 5.4].

When n is sufficiently large, by [15, Lemma 5.4], if an intersection point represents the fixed torsion Spin^c structure \mathfrak{s} , then its α_1 -component must lie in the winding region corresponding to α_1 . These intersection points are partitioned into two subsets according to whether they lie to the left or right side of a_1^* . Let L^+ (R^+) denote the subgroup of $\underline{CF}^+(Y, \mathfrak{s}; \Lambda_\omega)$ generated by the points which lie to the left (right) of a_1^* .

As proved in [15, Lemma 5.5 and Lemma 5.6], L^+ is a subcomplex of $\underline{CF}^+(Y, \mathfrak{s}; \Lambda_\omega)$, and R^+ is a quotient complex. We have a short exact sequence

$$0 \longrightarrow L^+ \longrightarrow \underline{CF}^+(Y, \mathfrak{s}; \Lambda_\omega) \longrightarrow R^+ \longrightarrow 0,$$

which gives rise to a long exact sequence:

$$\cdots \rightarrow H_{d+i}(L^+) \rightarrow \underline{HF}_{d+i}^+(Y, \mathfrak{s}; \Lambda_\omega) \longrightarrow H_{d+i}(R^+) \xrightarrow{\delta} H_{d+i-1}(L^+) \rightarrow \cdots.$$

By Corollary 3.4, for sufficiently large i , $\underline{HF}_{d+i}^+(Y, \mathfrak{s}; \Lambda_\omega) = 0$. It follows that for all sufficiently large N ,

$$\chi(\underline{HF}_{\leq d+N}^+(Y, \mathfrak{s}; \Lambda_\omega)) = \chi(H_{\leq d+N}(L^+)) + \chi(H_{\leq d+N+1}(R^+)). \quad (2)$$

On the other hand, define $f_1: R^+ \rightarrow L^+$ to be

$$f_1([x_i^+, i]) = [x_i^-, i - n_z(\phi)] t_\phi^{f_\phi \eta}.$$

Here ϕ is the disk connecting x_i^+ to x_i^- which is supported in the winding region corresponding to α_1 , and η is the cochain (representing ω) used to define $CF^\circ(Y; \Lambda_\omega)$. Then we have another short exact sequence:

$$0 \longrightarrow \ker f_1 \longrightarrow R^+ \xrightarrow{f_1} L^+ \longrightarrow 0,$$

which induces a long exact sequence:

$$\cdots \rightarrow H_{d+i}(\ker f_1) \longrightarrow H_{d+i}(R^+) \xrightarrow{f_1} H_{d+i-1}(L^+) \longrightarrow H_{d+i-1}(\ker f_1) \rightarrow \cdots.$$

It is proved in [15, Section 5.2] that $\ker f_1$ is a finite dimensional graded vector space over Λ , and has Euler characteristic $\chi(\ker f_1) = \pm T(Y, \mathfrak{s})$. So for all sufficiently large N ,

$$\chi(\ker f_1) = \chi(H_{\leq d+N}(L^+)) + \chi(H_{\leq d+N+1}(R^+)). \quad (3)$$

Combining equations (2), (3) and Corollary 3.4, we obtain the desired result. \square

Having proved Proposition 3.2, we can use the same argument as in [13, Section 3] to prove the following homological version of Theorem 1.2.

Proposition 3.5. *Suppose Y is a closed 3-manifold, $F \subset Y$ is an embedded torus. Let M be the 3-manifold obtained by cutting Y open along F . The two boundary components of M are denoted by F_-, F_+ . If there is a cohomology class $\omega \in H^2(Y; \mathbb{Z})$ such that $\omega([F]) \neq 0$ and $\underline{HF}^+(Y; \Lambda_\omega) = \Lambda$, then M is a homology product, namely, the two maps*

$$i_{\pm*}: H_*(F_{\pm}) \rightarrow H_*(M)$$

are isomorphisms.

4 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. We will essentially follow Ghigini's argument in [6], with little modifications when necessary.

Remark 4.1. Before we get into the proof, we make a remark on the smoothness of foliations. In [5], the foliations constructed are smooth, except possibly along torus components of the given taut surface. In the proof of [6, Theorem 3.8], one also modifies a foliation further by replacing a compact leaf F with an $F \times I$, which is foliated by $F \times t$'s. The new foliation may not be smooth if F is a torus. However, by [3, Proposition 2.9.4], the new foliation can be approximated in C^0 -topology by smooth weakly semi-fillable (hence weakly fillable by [2, 4]) contact structures. Hence one can run the now standard argument as in [10, Section 41] and [19] to get the nontriviality of the corresponding Heegaard Floer homology.

Lemma 4.2. *Conditions are as in Theorem 1.2. Cut Y open along F , the resulting manifold M has two boundary components F_+, F_- . Suppose $c \subset F_+$ is an essential simple closed curve, then there exists an embedded annulus $A \subset M$, such that c is one component of ∂A , and the other component of ∂A lies on F_- .*

Proof. First notice that M is a homological product by Proposition 3.5. We can glue the two boundary components of M by a homeomorphism ψ to get a new manifold Y_ψ with $b_1(Y_\psi) = 1$. Let $\gamma \subset Y - K$ be a closed curve which is Poincaré dual to ω . Then γ also lies in Y_ψ , we still denote its Poincaré dual in Y_ψ by ω , and we have $\omega([F]) \neq 0$ in the new manifold Y_ψ . By Corollary 2.3,

$$\underline{HF}^+(Y_\psi; \Lambda_\omega) \cong \underline{HF}^+(Y; \Lambda_\omega) \cong \Lambda.$$

Y_ψ satisfies all the conditions in Theorem 1.2, so we can work with Y_ψ instead of Y .

From now on we assume $b_1(Y) = 1$. We also assume that $\omega([F]) > 0$, otherwise we can change the orientation of F .

If the conclusion of the lemma does not hold, suppose $c = c_+ \subset F_+$ is an essential simple closed curve such that there does not exist an annulus A as in the statement of the lemma. Since M is a homology product, we can find a simple closed curve $c_- \subset F_-$ homologous to c_+ in M . We fix an arc $\delta \subset M$ connecting F_- to F_+ . Let $\mathcal{S}_m(+c)$ be the set of properly embedded surfaces $S \subset M$ such that $\partial S = (-c_-) \cup c_+$ and the algebraic intersection number of S with δ is m . $\mathcal{S}_m(+c) \neq \emptyset$ since M is a homology product.

For any surface $S \in \mathcal{S}_m$, its norm $x(S) > 0$. Otherwise one component of S must be an annulus A connecting c_- to c_+ , which contradicts our assumption.

By [12, Lemma 6.4], when m is sufficiently large, there is a connected surface $S_1 \in \mathcal{S}_m(+c)$ which gives a taut decomposition of M . If we reverse the orientation of c , when n is sufficiently large, as before there is $S_2 \in \mathcal{S}_n(-c)$ which gives a taut decomposition of M . As in [5], using these two decompositions, one can then construct two taut foliations $\mathcal{G}_1, \mathcal{G}_2$ of M , such that F_-, F_+ are leaves of them. These two foliations are glued to get two taut foliations $\mathcal{F}_1, \mathcal{F}_2$ of Y , such that F is a leaf of them. Suppose S is a surface in $\mathcal{S}_0(+c)$, then $-S \in \mathcal{S}_0(-c)$. We have

$$\begin{aligned} e(\mathcal{F}_1, S) &= e(\mathcal{F}_1, S_1 - mF) = \chi(S_1) - m \cdot 0 < 0, \\ e(\mathcal{F}_2, -S) &= e(\mathcal{F}_2, S_2 - nF) = \chi(S_2) - n \cdot 0 < 0, \end{aligned}$$

where $e(\mathcal{F}, S)$ is defined in [6, Definition 3.7]. Thus we conclude that

$$e(\mathcal{F}_1, S) \neq e(\mathcal{F}_2, S). \quad (4)$$

Choose a diffeomorphism $\phi: F_+ \rightarrow F_-$ such that $\phi(c_+) = c_-$. Let Y_ϕ be the 3-manifold obtained from M by gluing F_+ to F_- by ϕ . Decompose ϕ as a product of positive Dehn twists along non-separating curves $\{c_1, \dots, c_k\}$ on F , then Y_ϕ is obtained from Y by doing (-1) -surgeries along these curves. Let $-W$ be the cobordism obtained by adding 2-handles to $Y \times I$ along these curves with -1 framing. As in the beginning of this proof, ω also denotes an element in $H^2(Y_\phi; \mathbb{Z})$.

Since $\omega([F]) \neq 0$, by Corollary 2.3, the map

$$\underline{F}_{-W; \text{PD}(\gamma \times I)}^+ : \underline{HF}^+(Y; \Lambda_\omega) \rightarrow \underline{HF}^+(Y_\phi; \Lambda_\omega)$$

induced by the cobordism W is an isomorphism.

By Remark 4.1, one can approximate the foliations $\mathcal{F}_1, \mathcal{F}_2$ on $-Y$ by smooth weakly fillable contact structures ξ_1, ξ_2 on $-Y$. We can realize the above curves $\{c_1, \dots, c_k\}$ to be Legendrian knots in both ξ_1 and ξ_2 . Let ξ'_i be the contact structure on $-Y_\phi$ obtained from $(-Y, \xi_i)$ by doing $(+1)$ -contact surgeries along these Legendrian knots. By Proposition 2.8

$$\underline{F}_{-W; \text{PD}(\gamma \times I)}^+(c^+(-Y, \xi_i; \Lambda_\omega)) = c^+(-Y_\phi, \xi'_i; \Lambda_\omega)$$

for $i = 1, 2$. The hypothesis $b_1(Y) = 1$, $\omega([F]) > 0$, the fact that ξ_i is weakly fillable and Corollary 2.7 force $c^+(-Y, \xi_i; \Lambda_\omega) \neq 0$ for $i = 1, 2$. Since the map $\underline{F}_{-W; \text{PD}(\gamma \times I)}^+$ is an isomorphism as stated above,

$$c^+(-Y_\phi, \xi'_i; \Lambda_\omega) = \underline{F}_{-W; \text{PD}(\gamma \times I)}^+(c^+(-Y, \xi_i; \Lambda_\omega)) \neq 0$$

for $i = 1, 2$.

Let \overline{S} be the closed surface in Y_ϕ obtained by gluing the two boundary components of S together. As in [6, Lemma 3.10], (4) implies that

$$\langle c_1(\xi'_1), [\overline{S}] \rangle \neq \langle c_1(\xi'_2), [\overline{S}] \rangle,$$

so the Spin^c structures $\mathfrak{s}_{\xi'_1}$ and $\mathfrak{s}_{\xi'_2}$ are different. Therefore the two elements

$$c^+(-Y_\phi, \xi'_1; \Lambda_\omega) \in \underline{HF}^+(-Y_\phi, \mathfrak{s}_{\xi'_1}; \Lambda_\omega)$$

and

$$c^+(-Y_\phi, \xi'_2; \Lambda_\omega) \in \underline{HF}^+(-Y_\phi, \mathfrak{s}_{\xi'_2}; \Lambda_\omega)$$

are linearly independent. Hence $c^+(-Y, \xi_1; \Lambda_\omega)$ and $c^+(-Y, \xi_2; \Lambda_\omega)$ are also linearly independent. We hence get a contradiction to the assumption that $\underline{HF}^+(Y; \Lambda_\omega) = \Lambda$. \square

Proof of Theorem 1.2. Cut Y open along F , we get a compact manifold M . By Lemma 4.2, we can find two embedded annuli $A_1, A_2 \subset M$, each has one boundary component on F_- and the other boundary component on F_+ , and $\partial A_1 \cap \partial A_2$ consists of two points lying in F_- and F_+ , respectively. A further isotopy will ensure that $A_1 \cap A_2$ consists of exactly one arc, so a regular neighborhood of $A_1 \cup A_2$ in M is homeomorphic to $(T^2 - D^2) \times I$. Now the irreducibility of Y implies that $M = T^2 \times I$. This finishes the proof. \square

5 Proof of Theorem 1.4

We turn our attention to the proof of Theorem 1.4. We will use the following lemmas, which are very similar in spirit.

Lemma 5.1. *Let A, B be \mathbb{Z} -graded abelian groups such that $A_{(i)}$ and $B_{(i)}$ are finitely generated for each $i \in \mathbb{Z}$. Let $h, v: A \rightarrow B$ be homogeneous homomorphisms of the same degree d . Suppose h has a right inverse ι , i.e. $h \circ \iota = \text{Id}_B$. (Hence the degree of ι is $-d$.) Then the map*

$$h + tv: A \otimes \Lambda \rightarrow B \otimes \Lambda$$

is surjective and $\ker(h + tv) \cong \ker(h) \otimes \Lambda$.

Proof. Define a map

$$P = \sum_{j=0}^{\infty} \iota \circ (v\iota)^j (-t)^j.$$

It is well-defined by the fact that the composition $v\iota$ is of degree 0 and the assumption that $B_{(i)}$ is finitely generated for each $i \in \mathbb{Z}$. Clearly

$$(h + tv) \circ P = \text{Id}_B.$$

So $h + tv$ is surjective. Define a map $F: \ker(h) \otimes \Lambda \rightarrow \ker(h + tv)$ by

$$F(a) = \sum_{i=0}^{\infty} (-t\iota \circ v)^i(a).$$

It has a two sided inverse $G: \ker(h + tv) \rightarrow \ker(h) \otimes \Lambda$ defined by

$$G(b) = (1 + t\iota \circ v)b.$$

F and G are also well-defined. They define an isomorphism between $\ker(h + tv)$ and $\ker(h) \otimes \Lambda$. \square

Lemma 5.2. *Let A, B be \mathbb{Z} -graded abelian groups as in the above lemma such that their gradings are bounded from below. Suppose $f_1: A \rightarrow B$ and $f_2: A \rightarrow B$ are homomorphisms such that f_1 is homogeneous, and for every homogeneous element $a \in A$ each term in $f_2(a)$ has grading strictly lower than $f_1(a)$. If f_1 is surjective, then $f_1 + f_2$ is also surjective. Furthermore, $\ker(f_1 + f_2) \cong \ker(f_1)$.*

Proof. See [15, Proposition 5.8]. \square

Proof of Theorem 1.4. If the zero surgery $Y_0(K)$ is a torus bundle, take $\omega = \text{PD}(\mu) \in H^2(Y_0(K); \mathbb{Z})$ be the Poincaré dual of the meridian of K . By Theorem 1.1,

$$\underline{HF}^+(Y_0(K); \Lambda_\omega) \cong \Lambda,$$

and is supported in a single torsion Spin^c structure \mathfrak{s}_0 . For every $m > 0$, \mathfrak{s}_0 induces unique Spin^c structures on Y and $Y_m(K)$, respectively. We denote these Spin^c structures by $\mathfrak{s}, \mathfrak{s}_m$. According to Theorem 2.2 we have the following exact sequence:

$$\begin{array}{ccc} \underline{HF}^+(Y_0(K), \mathfrak{s}_0; \Lambda_\omega) & \longrightarrow & HF^+(Y_m(K), \mathfrak{s}_m) \otimes \Lambda \\ & \nwarrow & \nearrow F^+ \\ & HF^+(Y, \mathfrak{s}) \otimes \Lambda, & \end{array} \quad (5)$$

where the map F^+ is induced by the cobordism $W'_m: Y_m(K) \rightarrow Y$,

$$F^+ = \sum_{\{\mathfrak{t} \in \text{Spin}^c(W'_m), \mathfrak{t}|_Y = \mathfrak{s}, \mathfrak{t}|_{Y_m} = \mathfrak{s}_m\}} \underline{F}_{W'_m(K), \mathfrak{t}; \text{PD}(\mu \times I)}^+.$$

Note W'_m is a cobordism between two rational homology 3-spheres, as in [19, Section 3.1], the above maps are related to the untwisted case by the formula

$$F^+ = \pm t^c \cdot \sum_{\{\mathfrak{t} \in \text{Spin}^c(W'_m), \mathfrak{t}|_Y = \mathfrak{s}, \mathfrak{t}|_{Y_m} = \mathfrak{s}_m\}} \underline{F}_{W'_m(K), \mathfrak{t}}^+ \cdot t^{\frac{\langle c_1(\mathfrak{t}) \cup \text{PD}[\mu \times I], [W'] \rangle}{2}}. \quad (6)$$

F^+ has two distinguished summands $\underline{F}_{W'_m, \mathfrak{r}_k; \text{PD}(\mu \times I)}^+$ and $\underline{F}_{W'_m, \mathfrak{y}_k; \text{PD}(\mu \times I)}^+$. A simple calculation shows they are homogeneous maps of the same degree, and their degree is strictly larger than the degree of any other summand of F^+ [21, Lemma 4.4].

Take m to be sufficiently large, by Theorem 2.4, the exact triangle (5) can be identified with

$$\begin{array}{ccc} \underline{HF}^+(Y_0(K), \mathfrak{s}_0; \Lambda_\omega) & \longrightarrow & H_*(C(\mathfrak{s})\{\max(i, j) \geq 0\}) \otimes \Lambda \\ & \nwarrow & \nearrow F^+ \\ & H_*(C(\mathfrak{s})\{i \geq 0\}) \otimes \Lambda & \end{array} \quad (7)$$

Under this identification, using equation (6), F^+ can be written as:

$$F^+ = h_* + tv_* + \text{lower degree summands}.$$

On the other hand there is a short exact sequence (see [17, Corollary 4.5]):

$$0 \rightarrow C(\mathfrak{s})\{i \geq 0 \text{ and } j < 0\} \rightarrow C(\mathfrak{s})\{i \geq 0 \text{ or } j \geq 0\} \xrightarrow{p} C(\mathfrak{s})\{j \geq 0\} \rightarrow 0.$$

It induces a long exact sequence. Since $C(\mathfrak{s})\{j \geq 0\}$ and $C(\mathfrak{s})\{i \geq 0\}$ are chain homotopy equivalent, we get:

$$\begin{array}{ccc} \widehat{HFK}(Y, K, \mathfrak{s}, -1) & \xrightarrow{\quad\quad\quad} & H_*(C(\mathfrak{s})\{\max(i, j) \geq 0\}) \\ & \nwarrow \quad \nearrow h_* & \\ & H_*(C(\mathfrak{s})\{i \geq 0\}) & \end{array} \quad (8)$$

By assumption Y is an L -space, so $H_*(C(\mathfrak{s})\{i \geq 0\}) \cong \mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]$. It follows that h_* is surjective since it is U -equivariant and is an isomorphism at sufficiently large gradings. Moreover, h_* has a right inverse since its image is a free abelian group.

Using Lemma 5.1 and 5.2, $h_* + tv_*$ and F^+ are surjective, and

$$\ker(F^+) \cong \ker(h_* + tv_*) \cong \ker(h_*) \otimes \Lambda.$$

From exact sequences (7),(8)

$$\underline{HF}^+(Y_0(K), \mathfrak{s}; \Lambda_\omega) \cong \ker(F^+), \quad \widehat{HFK}(Y, K, \mathfrak{s}, -1) \cong \ker(h_*).$$

Hence $\widehat{HFK}(Y, K, \mathfrak{s}, -1)$ has rank 1. The same argument shows that for any other Spin^c structure \mathfrak{s}' on Y , $\widehat{HFK}(Y, K, \mathfrak{s}', -1) \cong 0$. So

$$\widehat{HFK}(Y, K, [F], -1) = \bigoplus_{\mathfrak{r} \in \text{Spin}^c(Y)} \widehat{HFK}(Y, K, \mathfrak{r}, -1)$$

is of rank one, hence K is a fibered knot by [12]. \square

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